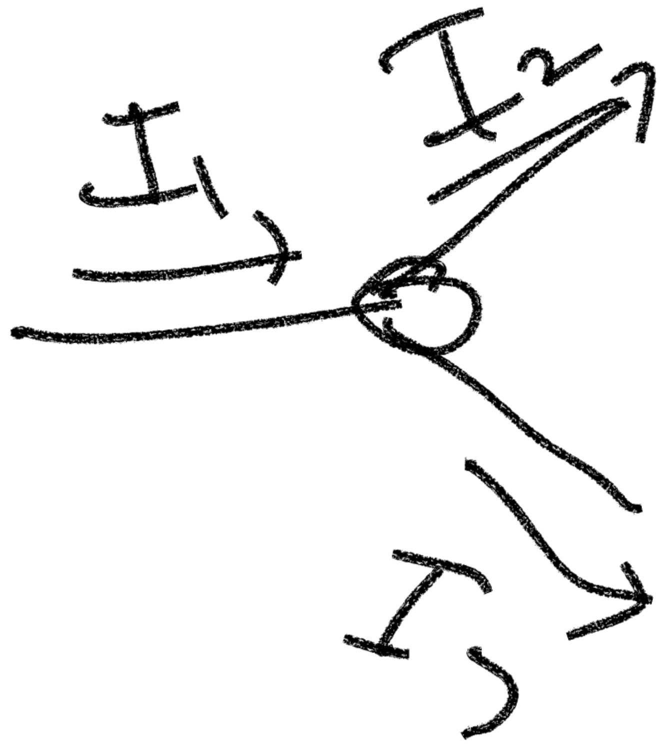
Lecture 13. Magnetostatics. Lorentz Force law

Last time: Circuits

* EMF = electroctromotive force

$$\mathcal{E} \equiv \frac{dW}{dq}$$ not a force, energy source

* Kirchhoff's rules:



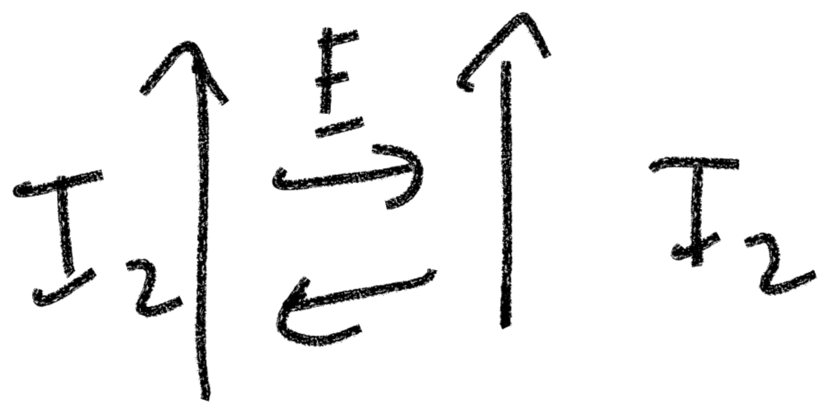
$$I_1 = I_2 + I_3$$

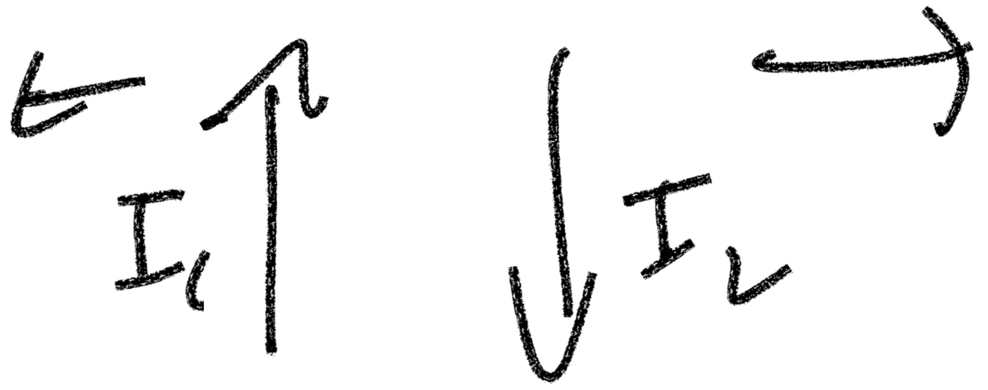
$$\sum_{\text{loop}} V = 0$$

loop rule

Today: Magnetism

Electricity & magnetism are related





Ampère's experiment

$\parallel$ I wires attract

Anti $\parallel$ wires repel

there $\exists$ a force due to a charge in motion.

Stationary charge $\rightarrow \underline{E}$

Moving charge $\rightarrow \underline{B}$

source of magnetic field

Charges in motion produce a magnetic field and there is a force felt on charges due to this

$$\boxed{\underline{F} = Q(\underline{v} \times \underline{B})}$$

Magnetic force on a charge Q moving at velocity $\underline{v}$

In the presence of both electric and magnetic fields

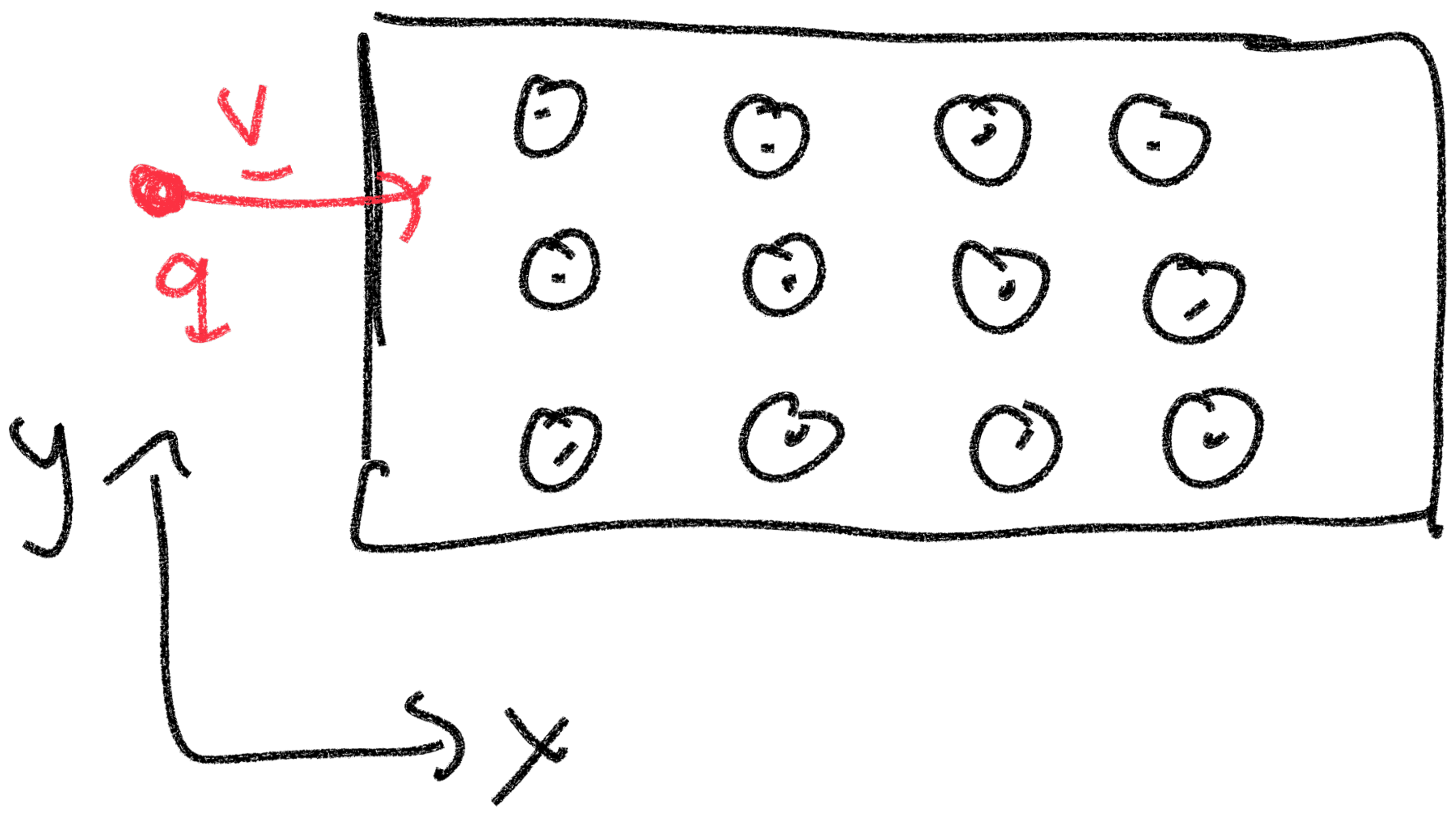
$$\boxed{\underline{F} = Q\left[\underline{E} + (\underline{v} \times \underline{B})\right]}$$

Lorentz's force law

$\underline{F} = q\,[\underline{E} + (\underline{v} \times \underline{B})]$ Lorentz's force law

* Empirical law that defines the magnetic field $\underline{B}$.
* $[B] = \dfrac{[F]}{qv} = \dfrac{N}{A \cdot m} \equiv$ Teslas
* Magnitude of this force is proportional to q and v
* Because $\underline{F} \propto \underline{v} \times \underline{B}$, when $\underline{v} \parallel \underline{B} \Rightarrow \underline{F}_{mag} = 0$
* If $\underline{v}$ makes an angle θ with $\underline{B}$, the direction of $\underline{F}$ is $\perp$ to the plane made by $\underline{v}$ and $\underline{B}$. And its magnitude is $\propto \sin\theta$
* Changing sign of q will reverse the direction of $\underline{F}$

Example: What is the trajectory of q?



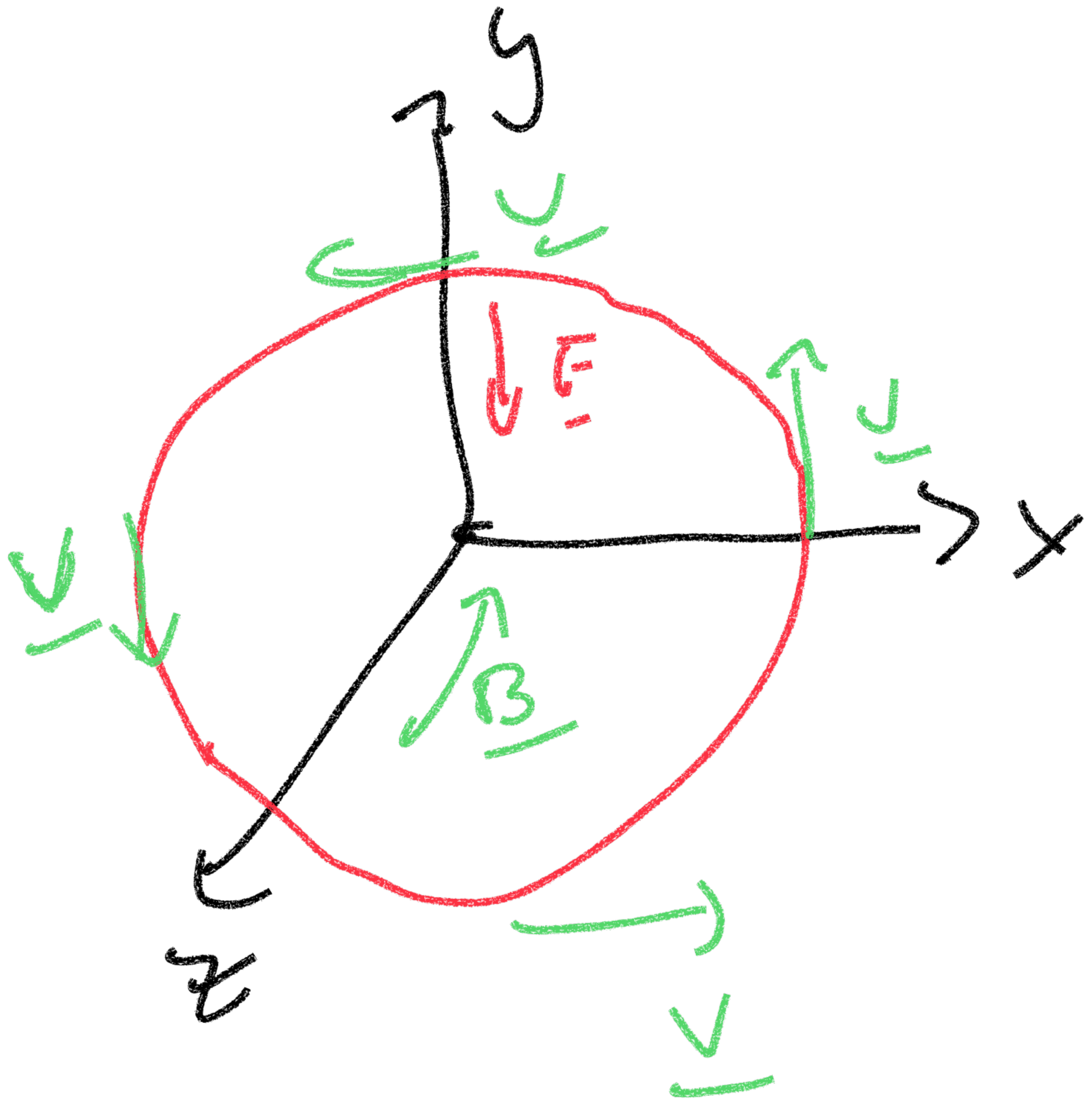
Particle of charge q, mass m
velocity $\underline{v} \parallel \hat{x}$
in a magnetic field $\underline{B} \parallel -\hat{z}$ axis (into the page.

$$\underline{F}_{mag} = q(\underline{v} \times \underline{B})$$

$\underline{B}$ and $\underline{v}$ are $\perp$

by definition $\underline{F}_{mag} \perp$ to the plane made by $\underline{v}$ & $\underline{B}$

$\Rightarrow$ $\underline{v}$, $\underline{B}$, $\underline{F}$ are all $\perp$



Particle moves in a counterclockwise circle.

Magnetic force has a fixed magnitude

$$|F_{mag}| = qvB$$

This allow it to sustain circular motion

$$qvB = \frac{mv^2}{R}$$ ← radius of the particle's trajectory.

$$\Rightarrow R = \frac{mv}{qB}$$

To complete one revolution:

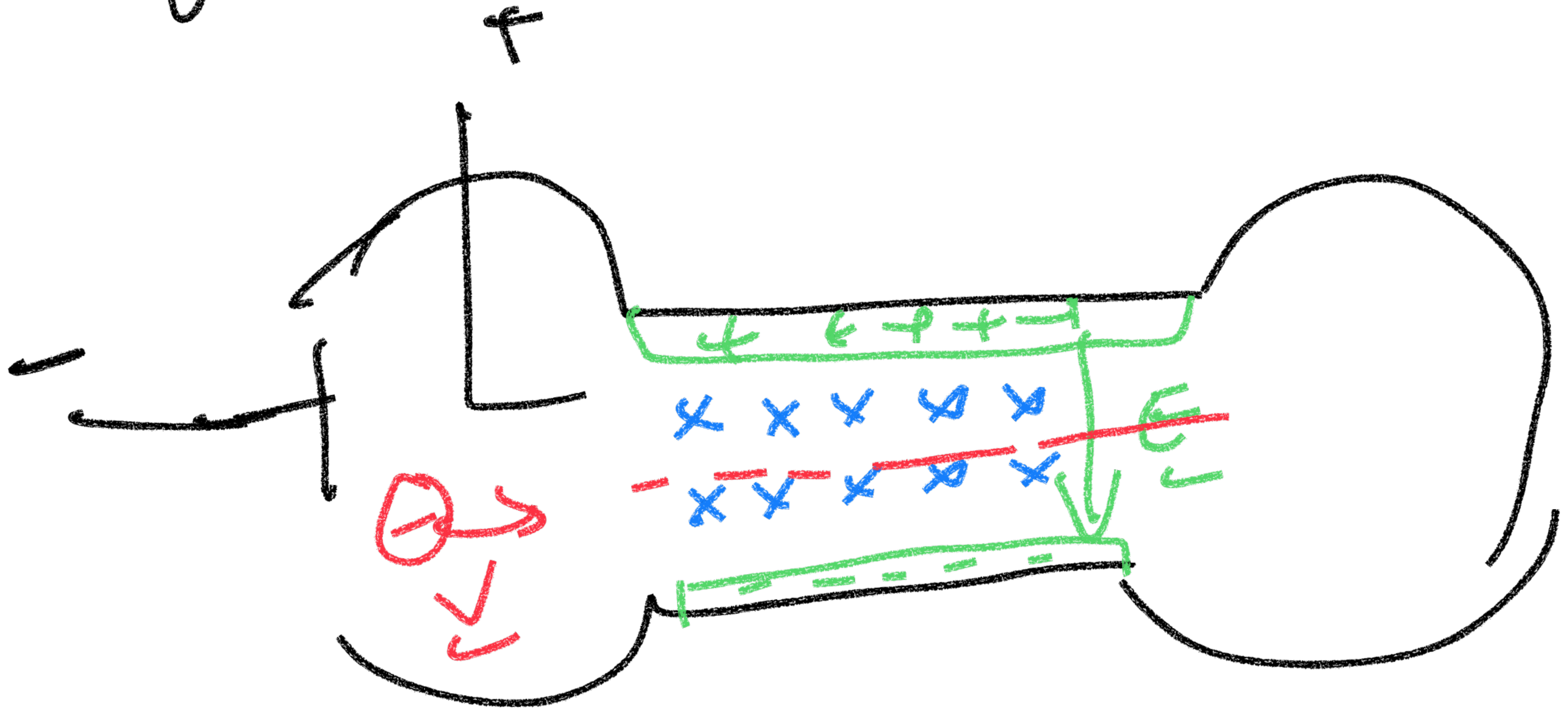
$$T = \frac{2\pi R}{v} = \frac{2\pi m v}{vqB} = \frac{2\pi m}{qB}$$

positive counterclockwise

negative clockwise

Example: JJ Thompson's experiment

Discovery of electron and measurements of e/m_e



$\ominus$ electron with charge $q = -e$

$\Delta V = V_A - V_c$ ← provides acc to the e^-

charge in potential energy

$\Delta U = W_{ext} = q \Delta V = -e \Delta V$

The kinetic energy gained is:

$\Delta K = -\Delta U = \frac{m v^2}{2} = e \Delta V \Rightarrow v = \left[\frac{2e \, \Delta V}{m}\right]^{1/2}$ speed of electrons

← potential difference

As the $\bar{e}$ passes through the region with $\underline{B}$ it will feel $\underline{E}$ and $\underline{B}$

If the particle moves in a straight trajectory:

$$|F_{el}| = |F_m|$$

$$\Rightarrow q\underline{E} = q\underline{v} \times \underline{B}$$

$$\Rightarrow eE = evB \Rightarrow v = \frac{E}{B} \quad (**)$$

Combining (*) and (**) we get:

$$\frac{E}{B} = \left[\frac{2e\Delta V}{m}\right]^{1/2} \Rightarrow \frac{e}{m} = \frac{E^2}{2\Delta V B^2} = 1.7588 \times 10^{11} \text{ C/kg}$$

Magnetic forces don't work.

* Moving a charge in an electric field requires work

$$W_{12} = -q \int_1^2 \underline{E} \cdot d\underline{s}$$

* How much work does it take to move a charge in a magnetic field?

$$dW = \underline{F} \cdot d\underline{s} = \underline{F} \cdot \underline{v}\, dt = q(\underline{v} \times \underline{B}) \cdot \underline{v}\, dt = 0$$

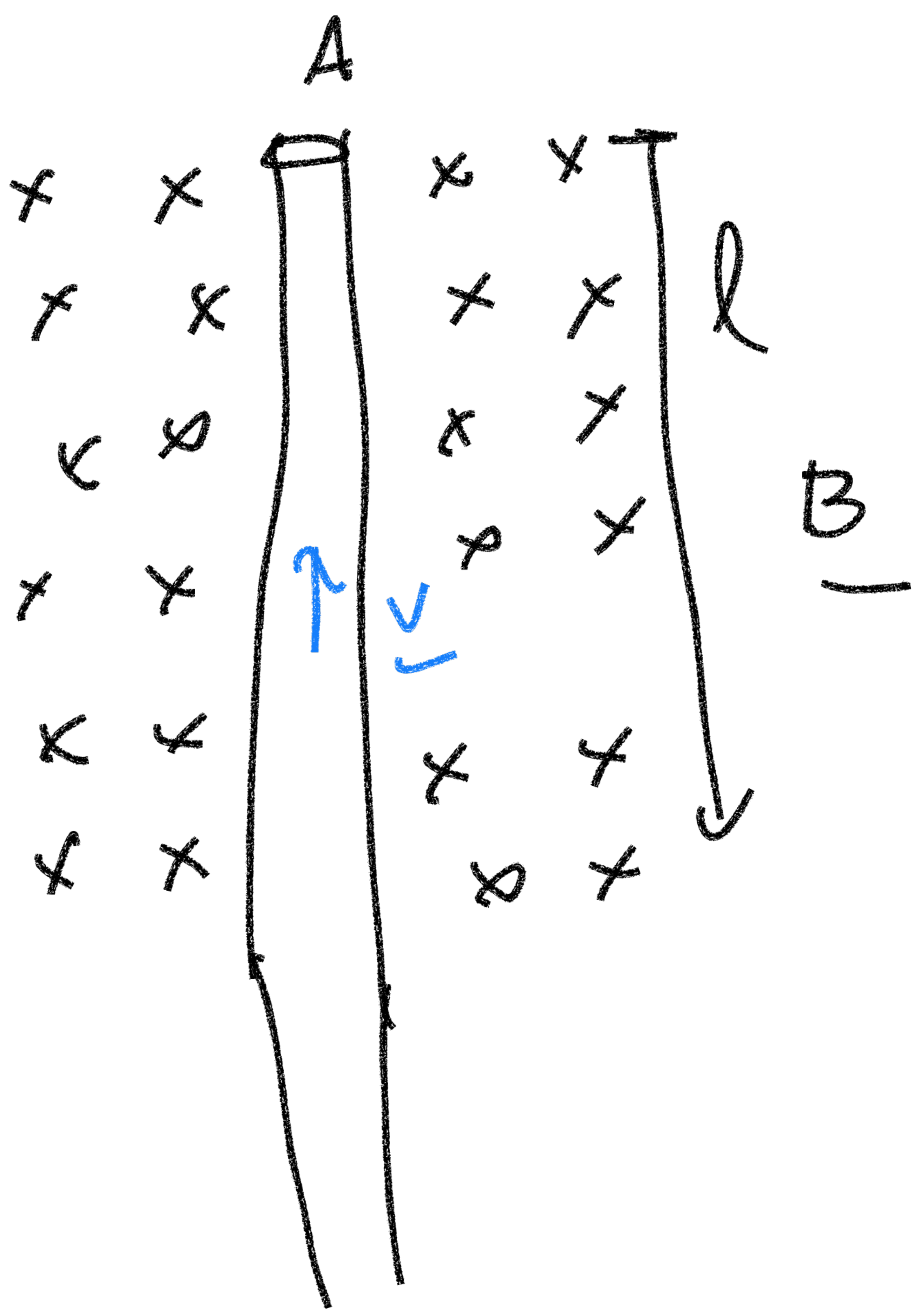
$(\underline{v} \times \underline{B})$ is $\perp$ to $\underline{v}$

$\Rightarrow (\underline{v} \times \underline{B}) \cdot \underline{v} = 0$

the dot product of 2 $\perp$ vectors is zero

$$\boxed{\text{Magnetic forces don't work}}$$

Magnetic force on a current carrying wire



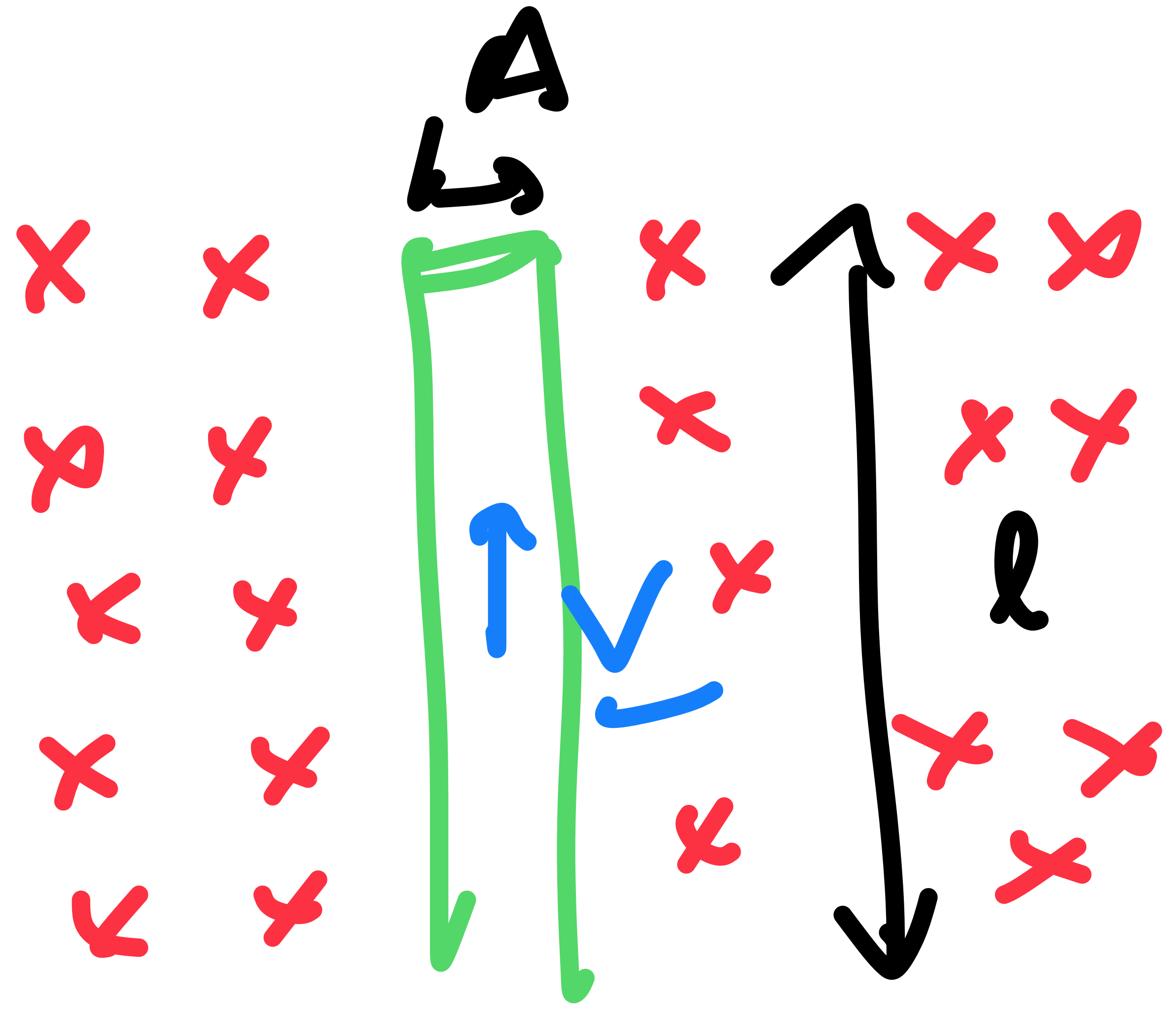
a current carrying wire experiences $\underline{F}$ due to $\underline{B}$.

What is the magnitude of this force

$A \equiv$ crossectional area of wire
$\underline{B} \equiv$ magnetic field (points into the page)
$l \equiv$ piece of wire considerd of length l

Steady current $\longrightarrow$ $\underline{v}$ is constant
average drift velocity

$Q_{tot} = q(nAl)$
$n \equiv$ # of charges/unit volume.



× B points into page

• out of the page

What is the force due to B?

$$Q_{tot} = q(nAl)$$

$$F_{mag} = Q_{tot}(\underline{v} \times \underline{B}) = \underbrace{qnA}_{I = qnAv}\, l(\underline{v} \times \underline{B}) = I(\underline{l} \times \underline{B})$$

$\underline{l}$ vector with length l and direction same as I

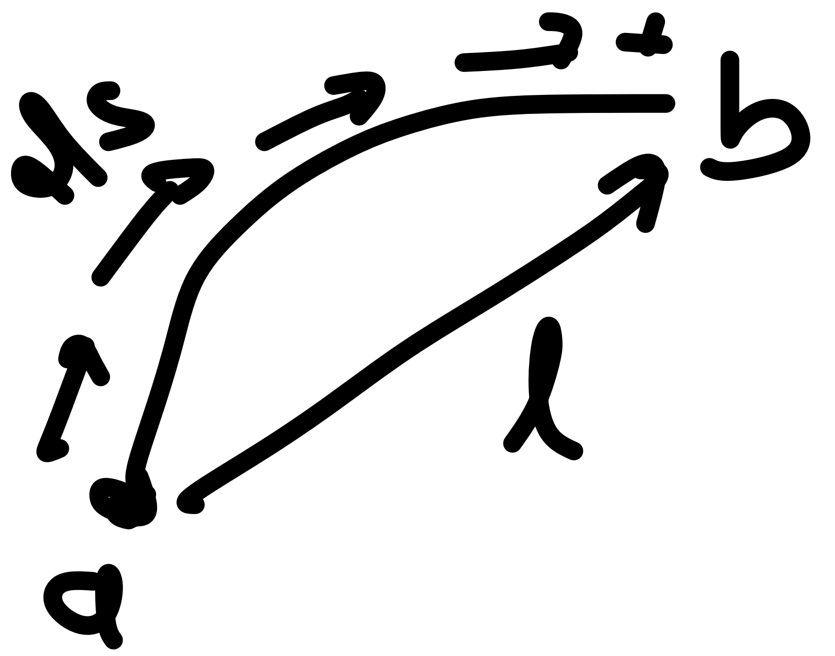
Infinitesimal wire of length $d\underline{s}$

The magnetic force acting on this element

$$\underline{F}_{mag} = I \, d\underline{s} \times \underline{B}$$

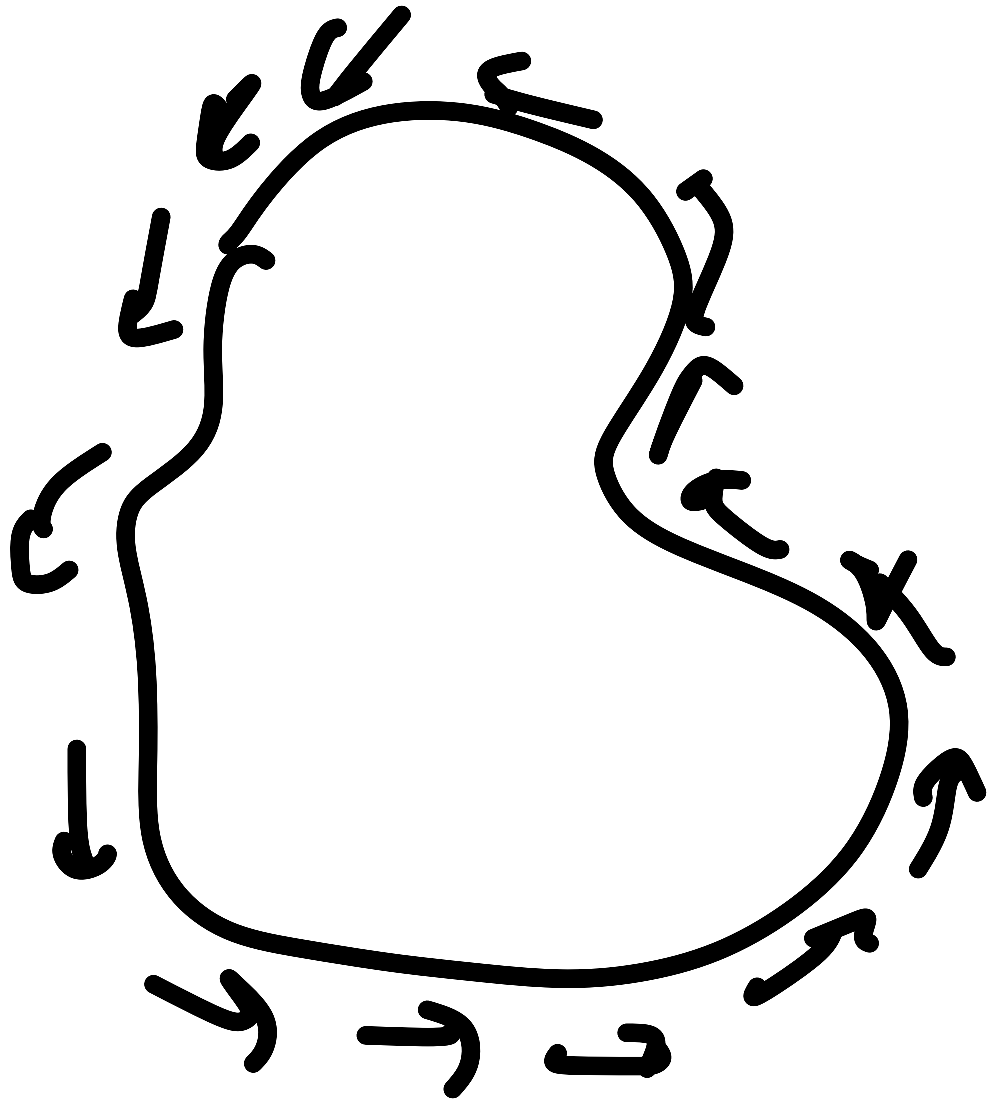
$$\Rightarrow \underline{F}_{mag} = I \int_a^b d\underline{s} \times \underline{B}$$ where a and b are the endpoints of the wire.

Consider the following wire as an example:



$$\underline{F}_{mag} = I \int_a^b d\underline{s} \times \underline{B} = I \, \underline{l} \times \underline{B}$$

Considering a closed loop of arbitrary shape:



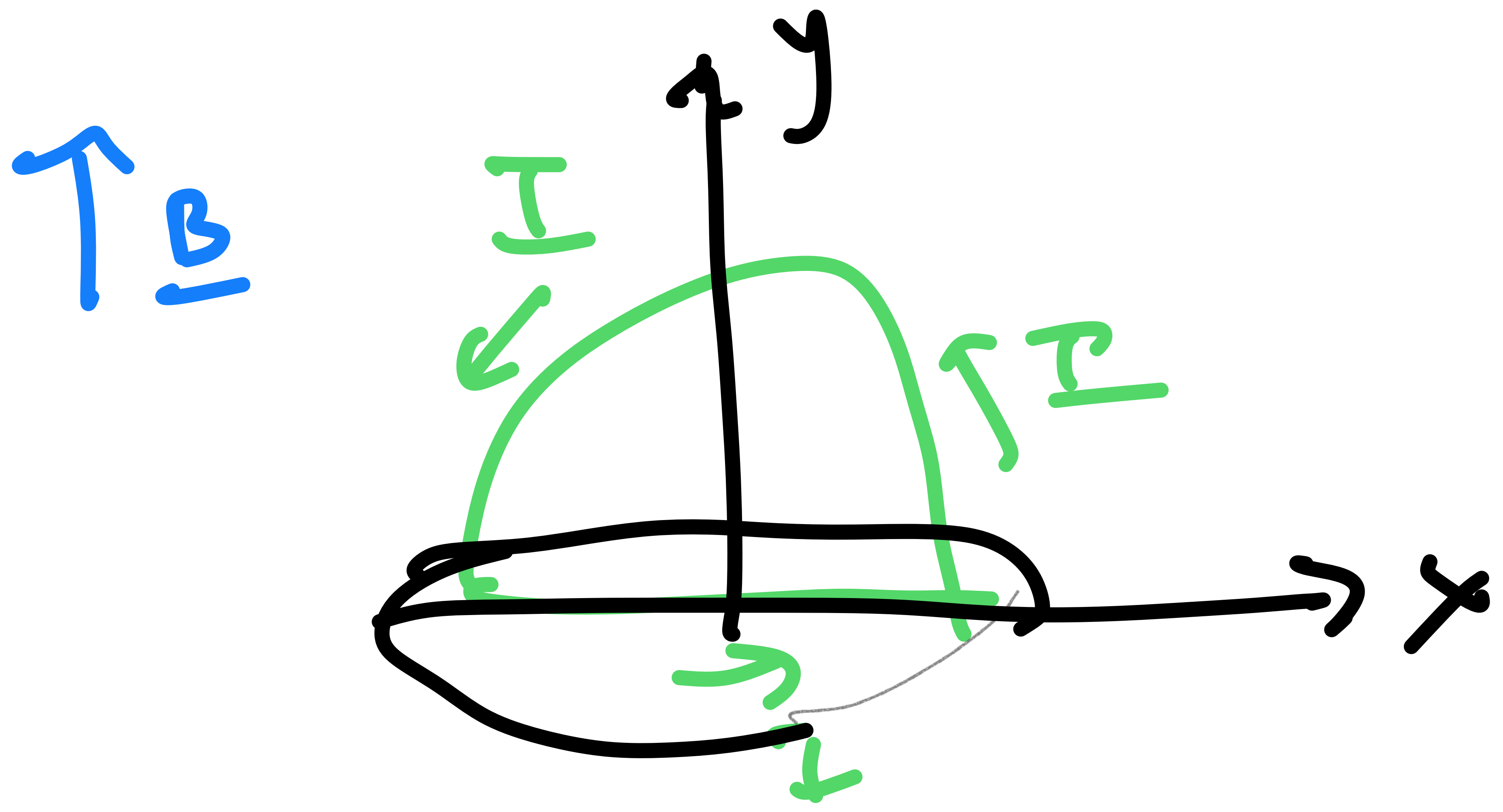
$$\underline{F}_{mag} = I \underbrace{\oint d\underline{s}}\, \times \underline{B}$$

$$\oint d\underline{s} = 0$$

$\therefore$ For any closed loop of current $\underline{F}_{mag} = 0$

Magnetic forces don't work.

Example. Magnetic force on a semicircular loop



I flowing counterclockwise

$\underline{B}$ uniform in $\hat{y}$

What is the magnetic force acting on the straight segment —

Let $\underline{B} = B\hat{j}$

Let $\underline{F}_1$ and $\underline{F}_2$ are the forces acting on ∩ and —

$$\underline{F}_{mag} = I\int_a^b d\underline{s} \times \underline{B}$$

$$\underline{F}_1 = I\,(2R)\hat{i} \times B\hat{j} = 2IRB\hat{k} \;,$$

radius of semicircle

Now we will evaluate $\underline{F}_2$:

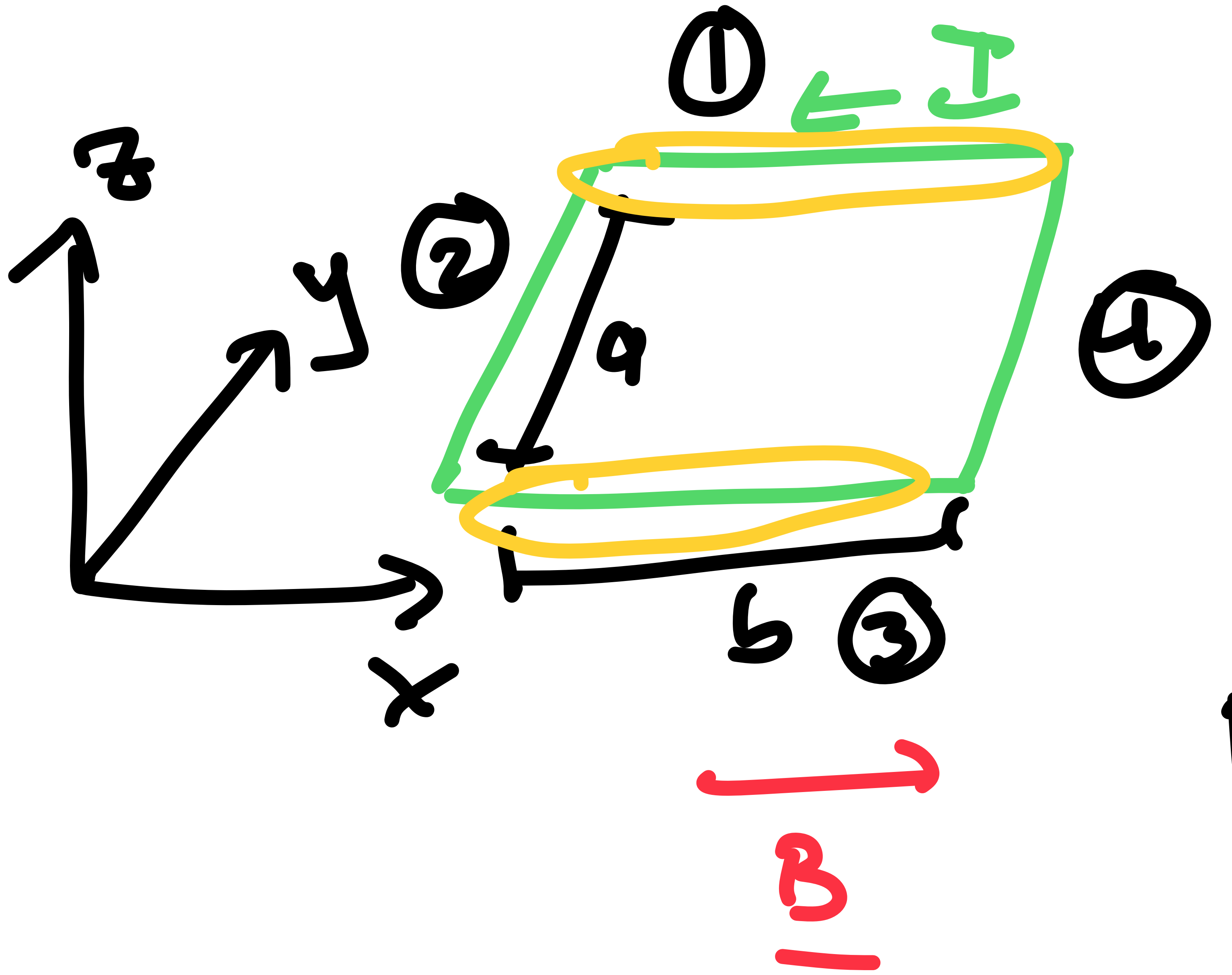
$$d\underline{s} = ds\,\hat{\theta} = R\,d\theta\,(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})$$ ← parametrizing an arc segment.

With this

$$\underline{F}_2 = -IBR\hat{k}\int_0^{\pi} \sin\theta\, d\theta = -2IBR\hat{k}$$

Torque on a current loop

What happens when we place a rectangular current-carrying loop in a uniform B?



loop has sides a and b

What is the magnitude of F experienced by each region in the loop?

For regions 1 & 3 $\quad F_{mag} = 0$

since $I \parallel B$

$$F_{mag} = I(\underline{l} \times \underline{B}) = 0$$

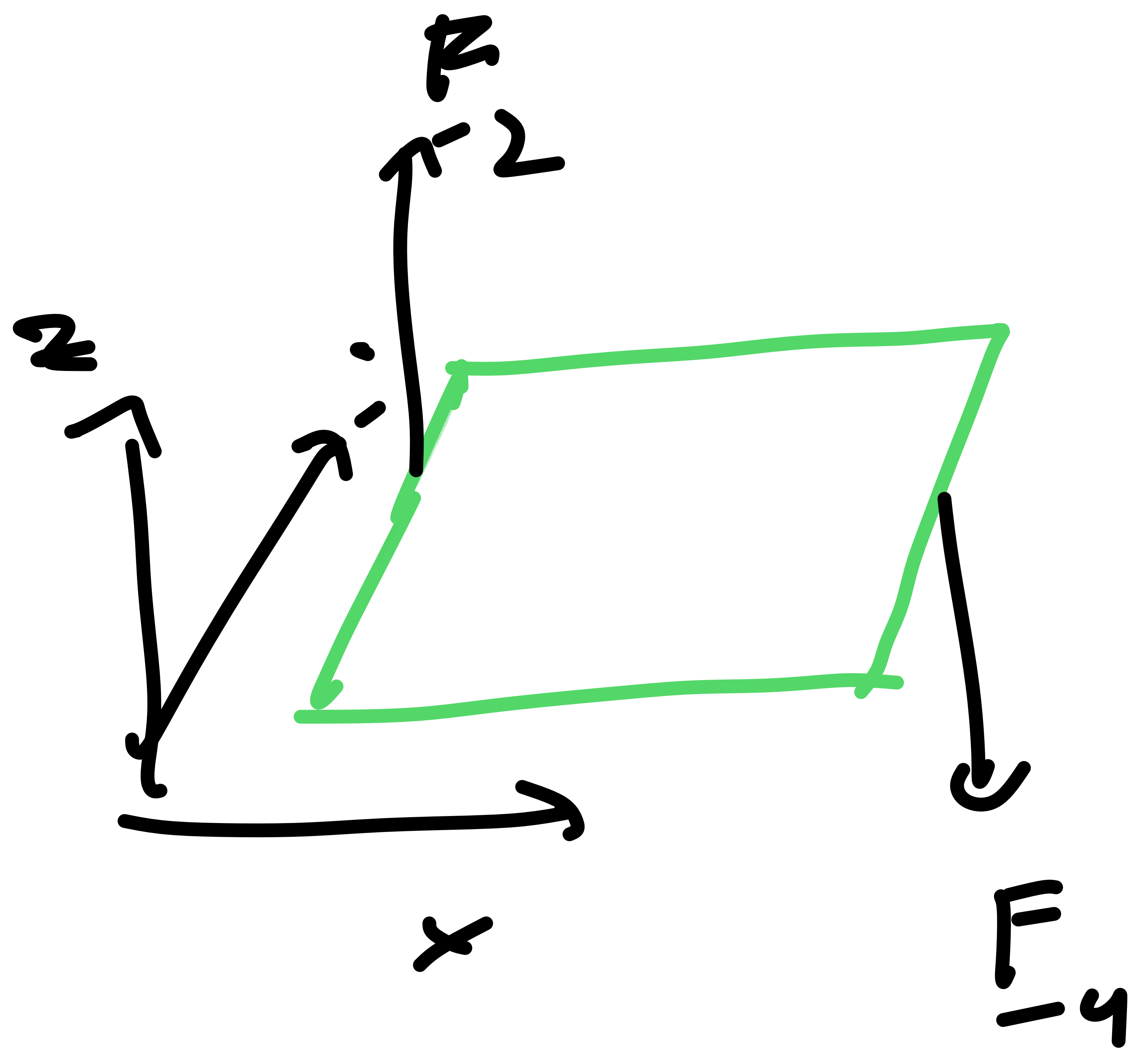
For regions 2 and 4 there is a magnetic force

$$\underline{F}_{mag} = I(\underline{l} \times \underline{B})$$

$$\underline{F}_2 = I(-a\hat{j}) \times B\hat{i} = IaB\hat{k}$$

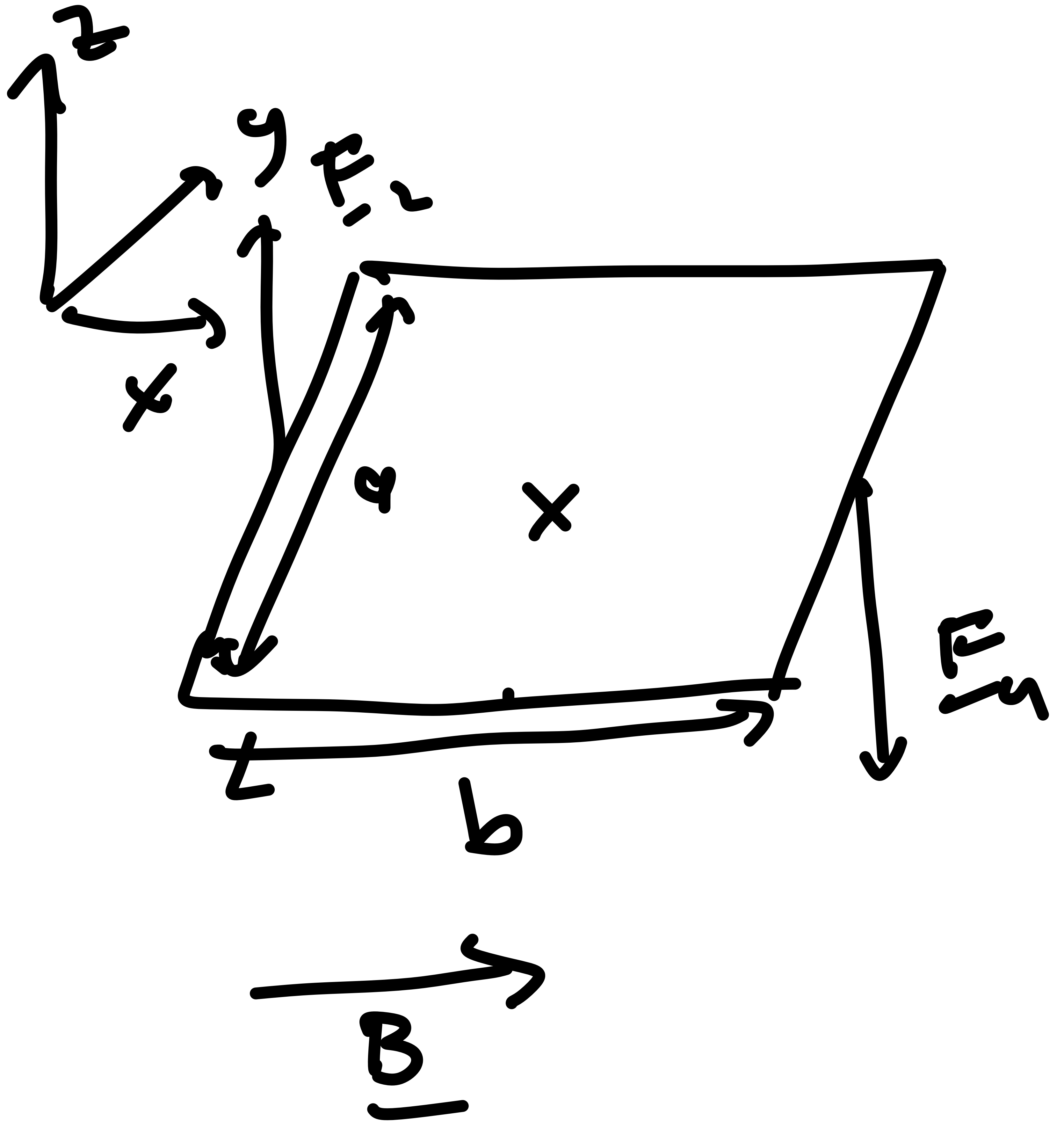
$$\underline{F}_4 = I(a\hat{j}) \times B\hat{i} = -IaB\hat{k}$$

$$\underline{F}_{net} = \underline{F}_1 + \underline{F}_2 + \underline{F}_3 + \underline{F}_4 = 0$$



$\underline{F}_2$ and $\underline{F}_4$ rotate the loop. They produce a torque: with respect to the center of the loop

$$\tau = \underline{r} \times \underline{F}$$



$$\tau = \left(-\frac{b}{2}\hat{i}\right) \times \underline{F}_2 + \left(\frac{b}{2}\hat{i}\right) \times \underline{F}_4$$

$$= \left(-\frac{b}{2}\hat{i}\right) \times \left(IaB\hat{k}\right) + \left(\frac{b}{2}\hat{i}\right) \times \left(-IaB\hat{k}\right)$$

subs the expressions for $\underline{F}_2$ and $\underline{F}_4$

$$= \left(\frac{IabB}{2} + \frac{IabB}{2}\right)\hat{j} = IAB\hat{j}$$

$A = ab$ area of the loop

$\therefore \ \underline{\tau} = IAB\hat{j}$

positive → rotation is clockwise.

We can rewrite this in terms of the area vector:

$$\underline{A} = A\hat{n}$$

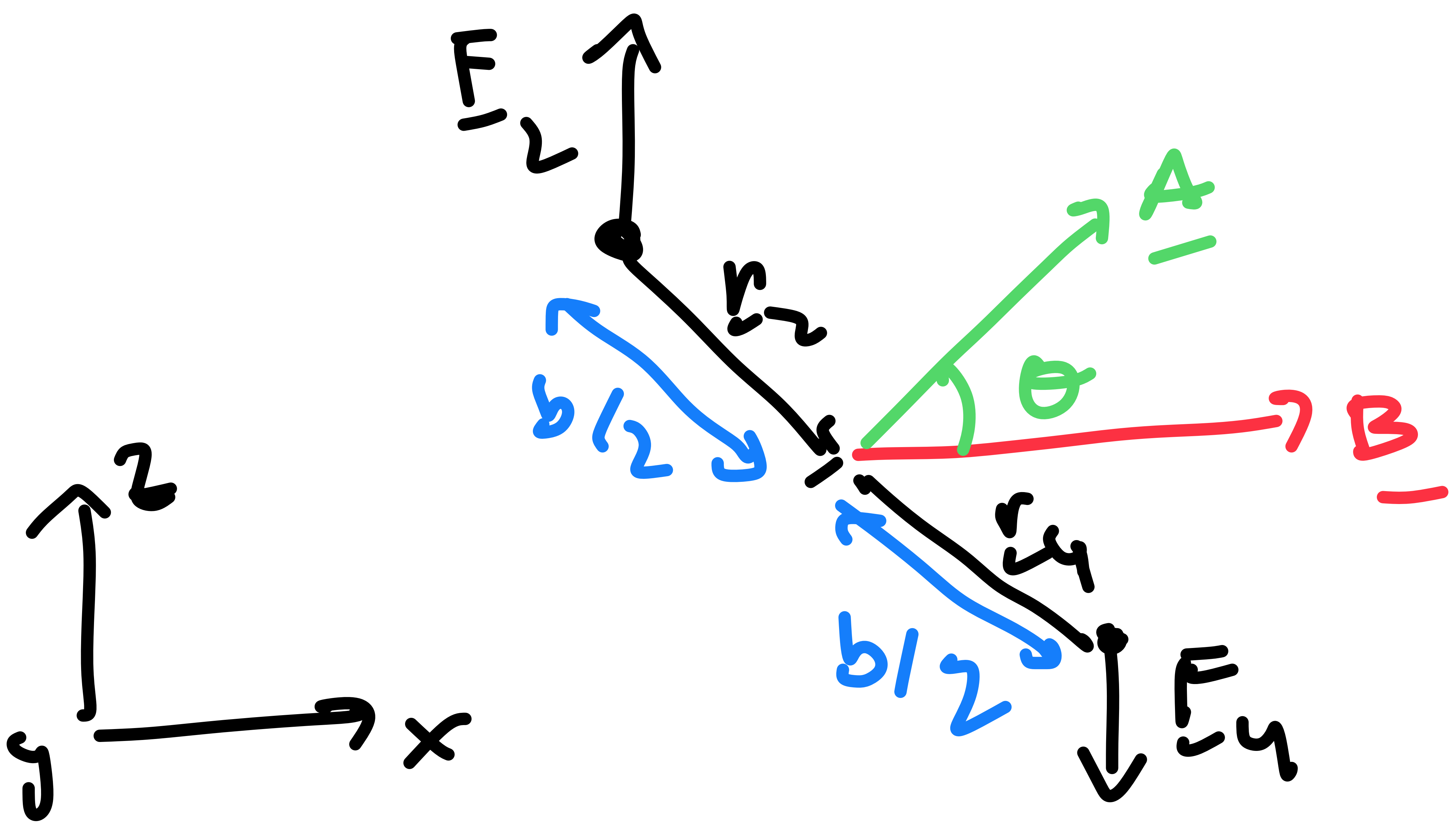
normal to the loop plane

In this case we have $\hat{n} = +\hat{k}$

$$\boxed{\underline{\tau} = I\underline{A} \times \underline{B}}$$

when $\underline{B}$ is // to the loop plane this τ will be maximum

Consider the general case where the loop is tilted with respect to $\underline{B}$:⟶ B is at angle with respect to the loop plane



$$* \; \underline{r}_2 = \frac{b}{2}\left(-\sin\theta\,\hat{i} + \cos\theta\,\hat{k}\right) = -\underline{r}_4$$

The net torque becomes:

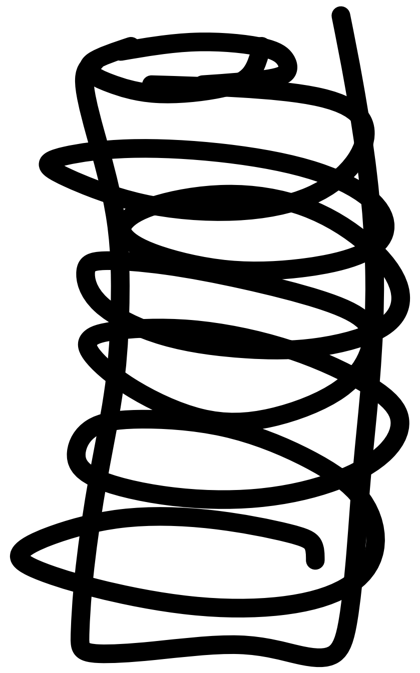
$$\tau = \underline{r}_2 \times \underline{F}_2 + \underline{r}_4 \times \underline{F}_4$$

$$= 2\underline{r}_2 \times \underline{F}_2$$

$$= \frac{2b}{2}\left(-\sin\theta\,\hat{i} + \cos\theta\,\hat{k}\right) \times \left(IaB\hat{k}\right)$$

$$= IabB\sin\theta\,\hat{j} = I\underline{A} \times \underline{B}$$

A loop with N "turns"



$$\tau = NIAB\sin\theta$$

$\underline{\mu} = NI\underline{A}$ the magnetic dipole moment $\underline{\mu}$

since $\underline{\mu}$ is in the same direction as $\underline{A}$:

$$\boxed{\underline{\tau} = \underline{\mu} \times \underline{B} \quad \text{where } \underline{\mu} = NI\underline{A}}$$